Published by Institute of Physics Publishing for SISSA/ISAS
Received: *May 4, 2002*
Revised: *December 9, 2002*
Accepted: *December 19, 2002*

# A realization of $N = 1$ $\mathcal{SW}(3/2, 2)$ algebras with Wolf spaces

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Abstract: We find out that some unitary minimal models of the $N = 1$ $\mathcal{SW}(3/2, 2)$ superconformal algebra can be realized as the level one coset models based on the Wolf spaces $\mathrm{SU}(n)/(\mathrm{SU}(n-2) \times \mathrm{SU}(2))$. We obtain the expression of the fermionic current with the conformal weight 5/2 in the algebra. Then, these models are twisted to give the topological conformal field theories.

Keywords: Topological Field Theories, Conformal Field Models in String Theory, Conformal and W Symmetry.

JHEP12(2002)059

## Contents



## 1. Introduction

Study of representations of super $\mathcal{W}$ algebras is one of fundamental problems in two-dimensional conformal field theories. The super $\mathcal{W}$ algebras would have their applications to provide an exact treatment of the superstring compactifications. Then we need a systematic understanding of the representations. At present such study is quite difficult to overcome, but Gepner and Noyvert [1] studied a construction of the unitary representations of the simplest $N = 1$ super $\mathcal{W}$ algebra, $\mathcal{SW}\,(3/2, 2)$ [2, 3]. This super $\mathcal{W}$ algebra was expected to play a fundamental role in the string compactifications on Spin(7) holonomy manifolds [4]. Meanwhile, there was much progress on the study of the string compactifications on exceptional holonomy manifolds [5]–[8].

It is well known that topological aspects of $N = 2$ superconformal models are often relevant to geometrical information on Calabi-Yau manifolds. Shatashvili and Vafa [4] suggested a construction of the topological twisting of the $c = 12$ $\mathcal{SW}\,(3/2, 2)$ algebra, and its implication to the conjectured mirror symmetry for Spin(7) holonomy manifolds. Their proposal is quite intriguing by itself because it hints generic structure of the topological counterparts of the $N = 1$ superconformal field theories based on the super $\mathcal{W}$ algebras. However, nobody has succeeded to provide an exact proof on the topological twisting. Motivated by this problem, we arrived at an observation on the $N = 1$ $\mathcal{SW}(3/2, 2)$ algebras with the central charges different from $c = 12$. The result arose from following two facts.

Let us begin with the result in [1] motivated by [9]. The $N = 1$ $\mathcal{SW}\,(3/2, 2)$ algebra consists of the two commuting Virasoro algebras, and the fermionic spin $3/2, 5/2$ currents. Then, we impose that one of the stress tensors should yield the $N = 0$ minimal model. This tightly restricts possible central charges of the algebra. The allowed values of the central charge are summarized in two discrete series:

$$c_p^{(1)} = 6 - \frac{18}{p+1}\,, \qquad c_p^{(2)} = 6 + \frac{18}{p}\,, \tag{1.1}$$

where $p$ is an integer such that $p \geq 3$. I.e., $3/2 \leq c_p^{(1)} \leq 6$ and $6 \leq c_p^{(2)} \leq 12$. At these values, we have the unitary minimal model which contain the representation of the $N = 0$ minimal model of $c = 1 - 6/p(p+1)$ as the subrepresentation of that of the $N = 1$ algebra. Then, the unitary model of $c_p^{(1)}$ has the spin $3/2$ supercurrent which is primary with the conformal weight $h_{1,2}$ under the Virasoro current of the $N=0$ minimal model. The conformal weights of the $N=0$ minimal model are given by $h_{r,s} = ([rp - s(1+p)]^2 - 1)/4p(1+p)$, ($r = 1\dots p$, $s = 1\dots p-1$). Here, the point is that the fermionic spin 5/2 current of the $N=1$ algebra is descendant with respect to the spin 3/2 supercurrent under the stress tensor of the $N=0$ minimal model. Furthermore, it is possible to reconstruct the whole $N=1$ $\mathcal{SW}(3/2,2)$ algebra from these facts in the $N=0$ minimal model. Closure of the operator product expansions of the $N=1$ algebra determines the value of the central charge of the $N=1$ algebra. Subsequently, the unitary representations of the $N=1$ algebra were determined.

The other ingredient is the result in [10]. We noticed that the central charge $c_p^{(1)}$ coincides with the central charge of the level one coset model based on the Wolf space $G/\left(H \times \mathrm{SU}(2)\right)$ [10]:

$$c = 6 - \frac{18}{g+1}\,, \tag{1.2}$$

where $g$ is the dual Coxeter number of the Lie group $G$.[1] In fact, we found out that the construction of the two Virasoro currents and the $N=1$ spin 3/2 superconformal current of the coset model is the same as that of the $N=1$ $\mathcal{SW}(3/2,2)$ algebra in [1]. This observation enabled us to write down the remaining fermionic spin 5/2 current of the $N=1$ $\mathcal{SW}(3/2,2)$ algebra through the coset realization. We will provide the explicit expression in the case with $G = \mathrm{SU}(n)$, but it should be straightforward to do with the other choices of $G$. Furthermore, we arrived at topological conformal field theories of the $N=1$ $\mathcal{SW}(3/2,2)$ algebra in the same way as [10, 11]. The BRST-exact expression of the twisted stress energy tensor of the topological theories follows from the $c=0$ coset models of $\left(G \times G\right)/G$. The BRST charge is given by the double contour integral of the two spin 3/2 supercurrents. We expect that the structure of the topological twisting might shed some light on the suggestion in [4].

Along these lines, we found out that the coset realization of the $N=1$ $\mathcal{SW}(3/2,2)$ algebras via the Wolf spaces $\mathrm{SU}(n)/(\mathrm{SU}(n-2)\times\mathrm{SU}(2))$. Then, we were led to the topological conformal field theories. We will state these observations in section 2. We will also include a relation of the $N=1$ coset models to the $N=2$ coset models. Section 3 will include conclusions and future problems arising from this note.

## 2. Construction of the $N=1$ $\mathcal{SW}(3/2,2)$ algebras

We begin with the level one $\mathrm{SU}(n)$ WZW theory using the Coulomb gas representation with the $n-1$ free scalar fields $\phi = (\phi_1, \phi_2, \dots, \phi_{n-1})$, and choose its Coulomb gas parameters as:

$$\alpha_+ = \sqrt{\frac{n+1}{n}}\,, \qquad \alpha_- = -\frac{1}{\alpha_+}\,, \qquad \alpha_0 = \alpha_+ + \alpha_- = \frac{1}{n\alpha_+}\,.$$

[1] The other series of the central charge $c_p^{(2)}$ may be obtained by substituting the level of the current algebra into negative values formally. We will not discuss this case in this note.

We also introduce the highest root and the Weyl vector of SU($n$):

$$\theta = \alpha_1 + \alpha_2 + \ldots \alpha_{n-1}, \qquad \rho_{\mathrm{SU}(n)} = \sum_{i=1}^{n-1} \frac{i(n-i)}{2}\alpha_i,$$

where $\alpha_i$ ($i = 1, \ldots, n-1$) are the simple roots of the SU($n$).

Let us define the SU(2) subgroup of SU($n$) by $E_\theta, E_{-\theta}$ and $[E_\theta, E_{-\theta}]$ where $E_\theta, E_{-\theta}$ are the lowering, raising operators. This gives the subgroup SU($n-2$) whose root vectors span the orthogonal subspace to $\theta$ in the root space of SU($n$). Then, we obtain the Wolf space SU($n$)/(SU($n-2$) $\times$ SU(2)) which is a symmetric space with a quaternionic structure [12]. Let us decompose the Weyl vector according to the Wolf space:

$$\rho_{\mathrm{SU}(2)} = -\frac{\theta}{2}, \qquad \rho_{\mathrm{SU}(n)} = \rho_{\mathrm{SU}(n-2)} + (1-n)\rho_{\mathrm{SU}(2)},$$

which leads to the $N = 1$ superconformal model of $c = 6 - 18/(n+1)$ [10]. Now, we will arrive at the $N = 1$ $\mathcal{SW}(3/2, 2)$ algebra. Let us recall that the generating currents of the algebra consist of the following four currents [2, 3]: the two bosonic spin 2 stress tensors $T_{N=1}, T_{\mathrm{SU}(2)}$, and the fermionic spin $3/2, 5/2$ currents $G, U$. The stress tensors are written down as

$$\begin{aligned} T_{N=1}(z) &= -\frac{1}{2}(\partial\phi)^2(z) + i\alpha_0\left(\rho_{\mathrm{SU}(n-2)} + \rho_{\mathrm{SU}(2)}\right)\partial^2\phi(z), \\ T_{\mathrm{SU}(2)}(z) &= -\frac{1}{2}\left(\sqrt{2}\rho_{\mathrm{SU}(2)}\partial\phi\right)^2(z) + \frac{i}{\sqrt{2}}\alpha_0\left(\sqrt{2}\rho_{\mathrm{SU}(2)}\partial^2\phi\right)(z). \end{aligned}$$

Comparing with the construction in [1], $T_{N=1}$ (resp. $T_{\mathrm{SU}(2)}$) is the stress tensor of the whole $N = 1$ $\mathcal{SW}(3/2, 2)$ algebra (resp. the $N = 0$ minimal model). The fermionic currents $G, U$ turn out to be

$$\begin{aligned} G(z) &= \psi_{\mathrm{SU}(2)}\left(\psi_{\mathrm{SU}(n-2),1} + \psi_{\mathrm{SU}(n-2),2}\right)(z), \\ &= e^{i\alpha_+\alpha_1\phi(z)} + e^{i\alpha_+\alpha_{n-1}\phi(z)}, \\ U(z) &= -\frac{2}{3}h_{1,2}\psi_{\mathrm{SU}(2)}\partial(\psi_{\mathrm{SU}(n-2),1} + \psi_{\mathrm{SU}(n-2),2})(z) + \\ &\quad + \left(1 - \frac{2}{3}h_{1,2}\right)\left(\partial\psi_{\mathrm{SU}(2)}\right)\left(\psi_{\mathrm{SU}(n-2),1} + \psi_{\mathrm{SU}(n-2),2}\right)(z), \end{aligned}$$

where $h_{1,2} = (n+3)/4n$ is the conformal weight in the Kac's table of the $N = 0$ minimal model. Here, we have introduced the following vertex operators

$$\begin{aligned} \psi_{\mathrm{SU}(2)} &= e^{-i\alpha_+\rho_{\mathrm{SU}(2)}\phi}, \\ \psi_{\mathrm{SU}(n-2),1} &= e^{i\alpha_+\frac{\alpha_1-\alpha_2-\ldots-\alpha_{n-1}}{2}\phi}, \qquad \psi_{\mathrm{SU}(n-2),2} = e^{-i\alpha_+\frac{-\alpha_1-\ldots-\alpha_{n-2}+\alpha_{n-1}}{2}\phi}. \end{aligned}$$

The complete expression of the currents of the $N = 1$ $\mathcal{SW}(3/2, 2)$ algebra of $c_n^{(1)} = 6 - 18/(n+1)$ through the Coulomb gas representation based on the Wolf space SU($n$)/(SU($n-2$) $\times$ SU(2)) is the main result in this note.

In fact, the coset models have their topological counterparts [10, 11]. The topological stress tensor is defined as

$$\begin{aligned} T_{c=0}(z) &= T_{N=1}(z) - i\alpha_0 n \rho_{\mathrm{SU}(2)} \partial^2\phi(z)\,, \\ &= -\frac{1}{2}(\partial\phi)^2(z) + i\alpha_0 \rho_{\mathrm{SU}(n)} \partial^2\phi(z)\,. \end{aligned}$$

This topological stress tensor can be written in the BRST-exact form

$$T_{c=0}(z) = \left\{ Q_{BRST},\, e^{-i\alpha_+\theta\phi}(z) \right\}.$$

Here, the BRST charge $Q_{BRST}$ is defined as the double contour integral of the two spin 3/2 superconformal currents $G$ using a suitable path of the integration

$$Q_{BRST} = \int\int dz\, dw\, G(z)G(w)\,,$$

and satisfies the nilpotency: $Q_{BRST}^2 = 0$. It is possible to prove this nilpotency by checking poles in the integral. The BRST-exactness of the topological stress tensor is also proven by inserting the screening charge $\int dz\, e^{i\alpha_+\alpha_i\phi(z)}$ in the integral. The conformal weight $h_{c=0}$ of the fermionic current $G$ (resp. $U$) under the topological stress tensor $T_{c=0}$ is an integer, i.e., one (resp. two). In [4], the fermionic spin 5/2 current $U(z)$ is suggested to be BRST-equivalent to the anti-ghost field $e^{-i\alpha_+\theta\phi(z)}$: $U(z) = e^{-i\alpha_+\theta\phi(z)} + \{Q_{BRST}, *\}$. We do not understand a role of the current $U(z)$ in the present topological model.

Finally, we discuss that the $N = 1$ superconformal coset models have a relation to the standard $N = 2$ superconformal coset models [13] (The same argument was given in [14].). Let us explain it with the Wolf space $\mathrm{SU}(4)/(\mathrm{SU}(2) \times \mathrm{SU}(2))$. There is the cyclic $\mathbf{Z}_4$ symmetry in the root system of the affine Lie algebra $\hat{su}(4)$: $(\alpha_0, \alpha_1, \alpha_2, \alpha_3) \to (\alpha_3, \alpha_0, \alpha_1, \alpha_2)$ (The reader should not confuse the root $\alpha_0$ with the Coulomb gas parameter.). Then, the $N = 1$ superconformal currents $T^{N=1}, G^{N=1}$ are mapped into the $N = 2$ superconformal currents $T^{N=2}, G^{\pm,N=2}$ of the coset model of the symmetric space $\mathrm{SU}(4)/(\mathrm{SU}(2)^2 \times \mathrm{U}(1))$:

$$\begin{aligned} T^{N=1}_{\mathrm{SU}(4)/(\mathrm{SU}(2)\times\mathrm{SU}(2))} &= -\frac{1}{2}(\partial\phi)^2 + i\alpha_0 \left( -\frac{\alpha_0+\alpha_2}{2} \right) \partial^2\phi\,, \\ &= -\frac{1}{2}(\partial\phi)^2 + i\alpha_0 \left( \frac{\alpha_1+\alpha_3}{2} \right) \partial^2\phi\,, \\ &:= T^{N=2}_{\mathrm{SU}(4)/(\mathrm{SU}(2)^2\times U(1))}\,, \end{aligned}$$

$$\begin{aligned} G^{N=1}_{\mathrm{SU}(4)/(\mathrm{SU}(2)\times\mathrm{SU}(2))} &= e^{i\alpha_+\alpha_0\phi} + e^{i\alpha_+\alpha_2\phi}\,, \\ &= e^{-i\alpha_+\theta\phi} + e^{i\alpha_+\alpha_2\phi}\,, \\ &:= G^{-,N=2}_{\mathrm{SU}(4)/(\mathrm{SU}(2)^2\times U(1))} + G^{+,N=2}_{\mathrm{SU}(4)/(\mathrm{SU}(2)^2\times U(1))}\,. \end{aligned}$$

Here, the part of the vertex operators with the imaginary root $\delta = \sum_{i=0}^{n-1} \alpha_i$ can be ignored in the operator product expansions of these currents due to the orthogonality: $(\delta, \alpha_i) = 0$. Further, it is possible to write down the U(1) current of the $N = 2$ model

$$J^{N=2} = 2i\alpha_0 \left( \rho_{\mathrm{SU}(4)} - \rho_{\mathrm{SU}(2)^2} \right).\partial\phi.$$

In this way, we arrive at the relation of the $N = 1$ coset models to the $N = 2$ coset models of the SU($n$) group with the SU(2) factor in the denominator. Here, we do not have a role of the fermionic current $U$ in the $N = 2$ coset models. Alternatively, it is straightforward to obtain another class of the $N = 2$ coset models. For example, to obtain the coset model of the symmetric space SU(4)/(SU(3) × U(1)), we delete the simple roots $\alpha_1, \alpha_2$ of SU(4) in the construction of [10].

## 3. Conclusion and outlook

We pointed out that the $N = 1$ superconformal $\mathcal{SW}(3/2, 2)$ algebras with the specific central charge are realized by the Wolf spaces SU($n$)/(SU($n-2$) × SU(2)). Then, we discussed some consequences of this observation: the existence of the topological conformal field theories and the relation to the $N = 2$ superconformal coset models. Through the realization, we confirmed the existence of the topological conformal field theories based on the $N = 1$ super $\mathcal{W}$ algebras.

Based on the result in this note, a lot of things should be considered:

1. It is desirable to analyze in detail the spectrum in the $N = 1$ superconformal coset models. We will dwell only on the states in the NS sector of the $c = 3/2$ ($p = n = 3$) model. The primary fields in the NS sector are labelled by the conformal weights $h, a$ under the stress tensors $T_{N=1}$, $T_{\mathrm{SU}(2)}$. Then, we arrive at the vertex operators $1$, $e^{i\alpha_0\alpha_1\phi}$, $e^{i\alpha_0(2\alpha_1+2\alpha_2)\phi}$. Here, we have imposed that the conformal weight $a$ should exist in the Kac's table of the Ising model of the stress tensor $T_{\mathrm{SU}(2)}$. We write these states as $|0, 0\rangle$, $|1/16, 1/16\rangle$, $|1/2, 0\rangle$ in the way that the state with the conformal weights $h, a$ is denoted by $|a, h-a\rangle$ (In fact, these states coincide with those of the minimal model of $c^{(1)}_{p=3} = 3/2$ in [1].). Meanwhile, when we move into the topological conformal field theory, these states turn out to be the BRST invariant states in the sense that $Q_{BRST}|\Psi\rangle = 0$ where $|\Psi\rangle$ is the state corresponding to the vertex operator. Subsequently, we should proceed to the highest weight states in the Ramond sector of the $N = 1$ coset model. Then, it would be very interesting to consider a geometrical interpretation of the spectrum through the Wolf space. It also might be of use to visit the gauged WZW model on the Wolf space [15].

2. It would be very interesting to clarify a relation between the present result and the result in [1]. The unitary spectrum in [1] has the continuous parts. The $N = 1$ coset model and the $\mathbf{Z}_2$ orbifold of the coset model of the $N = 2$ super $W_3$ algebra [9] should provide the special realization of the continuous spectra. Then, this might suggest a new kind of mirror symmetry because it is likely that the Wolf spaces and the orbifolds of the hermitian symmetric spaces are geometrically different even if the corresponding $N = 1$ superconformal field theories are the same.

3. Extension of the result into the Lie groups other than $G = \mathrm{SU}(n)$ is of interest (see [11]). Along the quaternionic structure of the Wolf spaces, it would be interesting to consider the extension with $G = \mathrm{Sp}(n)$. Then, it is natural to ask how to obtain



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the coset realization of the $N = 1$ super $\mathcal{W}$ algebra through a hamiltonian reduction along the well-known story in the $N = 2$ coset model [16, 3]. In a related direction, the $N = 1$ $\mathcal{SW}(3/2, 3/2, 2)$ algebra was studied using the $(\mathrm{SU}(2) \times \mathrm{SU}(2))\,/\,\mathrm{SU}(2)$ coset model motivated by an application to the string compactifications on $G_2$ holonomy manifolds [7]. The result might be understood through a hamiltonian reduction as in [17].

We hope to report further developments on these problems elsewhere.


## Acknowledgments

We thank T. Eguchi, T. Kawai, Y. Koga, K. Ito and A. Yamaguchi for helpful conservations. We are also grateful to D. Gepner and B. Noyvert for comments. Especially, we wish to thank H. Kanno for collaborative discussions and comments on the manuscript. Present research is supported by Japanese Society of the Promotion of Science under Post-doctorial Research Program (No. 0206911).


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